Model Solutions

Oxford Cambridge and RSA

Monday 20 May 2019 – Afternoon

AS Level Further Mathematics A

Y535/01 Additional Pure Mathematics

Time allowed: 1 hour 15 minutes

You must have:
- Printed Answer Booklet
- Formulae AS Level Further Mathematics A

You may use:
- a scientific or graphical calculator

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INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION
- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

Answer **all** the questions.

1 In decimal (base 10) form, the number N is 15 260.

(a) Express N in binary (base 2) form. **[1]**

(b) Using the binary form of N, show that N is divisible by 7. **[2]**

a) 11101110011100_2

b) $7_{10} = 111_2$

$\therefore 11101110011100_2$

$= 100010000100_2 \times 111_2$

$= 7 | N$

2 **(a)** The convergent sequence $\{a_n\}$ is defined by $a_0 = 1$ and $a_{n+1} = \sqrt{a_n} + \frac{4}{\sqrt{a_n}}$ for $n \geqslant 0$.

Calculate the limit of the sequence. **[1]**

(b) The convergent sequence $\{b_n\}$ is defined by $b_0 = 1$ and $b_{n+1} = \sqrt{b_n} + \frac{k}{\sqrt{b_n}}$ for $n \geqslant 0$, where k is a constant.

Determine the value of k for which the limit of the sequence is 9. **[3]**

a) Limit $= 4$ $\Rightarrow \left(a_{n+1} = a_n = \sqrt{4} + \frac{4}{\sqrt{4}} = 4\right)$

b) $b_{n+1} = b_n = 9$

$\therefore 9 = \sqrt{9} + \frac{k}{\sqrt{9}} = 3 + \frac{k}{3}$

$6 = \frac{k}{3}$ $\quad \underline{k = 18}$

3 The non-zero vectors $\mathbf{x}$ and $\mathbf{y}$ are such that $\mathbf{x} \times \mathbf{y} = \mathbf{0}$.

(a) Explain the geometrical significance of this statement. **[2]**

(b) Use your answer to part **(a)** to explain how the line equation $\mathbf{r} = \mathbf{a} + t\mathbf{d}$ can be written in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{d} = \mathbf{0}$. **[2]**

a) x and y are parallel.

$x \times y = xy\sin\theta u$ (where u is the unit vector)

$= 0$

since $x, y \neq 0$, $\sin\theta = 0$ $\therefore$ $\theta = 0$ or π and $x \parallel y$.

b) $r = a + td$

$r - a = td$

$(r-a) \parallel d = 0$

Then using part a, $(r-a) \times d = 0$

4 The sequence $\{u_n\}$ is defined by $u_1 = 1$ and $u_{n+1} = 2u_n + n^2$ for $n \geqslant 1$.

Determine u_n as a function of n. **[8]**

$u_{n+1} = 2u_n + n^2 \quad \Rightarrow u_{n+1} - 2u_n = n^2$

. Complementary Solution: $u_n = A \times 2^n$

. Particular Solution: try $u_n = an^2 + bn + c$

Using $u_{n+1} - 2u_n = n^2$:

$\therefore an^2 + 2an + a + bn + b + c - 2(an^2 + bn + c) = n^2$

Comparing coefficients:

n^2 coeff: $a - 2a = 1$

$-a = 1 \quad \Rightarrow a = -1$

n coeff: $2a + b - 2b = 0$

$$2a - b = 0$$

$$2(-1) - b = 0 \Rightarrow b = -2$$

constants: $a + b + c - 2c = 0$

$$a + b - c = 0$$

$$(-1) + (-2) - c = 0$$

$$c = -3$$

So Particular Solution: $u_n = -(n^2 + 2n + 3)$

General Solution: $u_n = A \times 2^n - (n^2 + 2n + 3)$

when $n = 1$

$$u_1 = 1 = 2A - (1^2 + 2(1) + 3)$$

$$1 = 2A - 6$$

$$7 = 2A$$

$$\frac{7}{2} = A$$

$$\therefore u_n = 7 \times 2^{n-1} - (n^2 + 2n + 3)$$

5 The tetrahedron T, shown below, has vertices at $O(0, 0, 0)$, $A(1, 2, 2)$, $B(2, 1, 2)$ and $C(2, 2, 1)$.

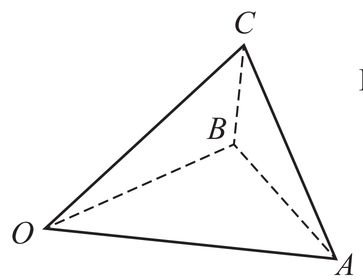


Diagram not drawn to scale

Show that the surface area of T is $\frac{1}{2}\sqrt{3}\left(1+\sqrt{51}\right)$. **[8]**

Area $\Delta OAB = \frac{1}{2}|a \times b|$

Using the cross product:

$$a \times b = \begin{pmatrix}1\\2\\2\end{pmatrix} \times \begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{vmatrix} = \begin{pmatrix}4-2\\-(2-4)\\1-4\end{pmatrix}$$

$$= \begin{pmatrix}2\\2\\-3\end{pmatrix}$$

$\therefore$ Area of ΔOAB $= \frac{1}{2}\sqrt{2^2+2^2+(-3)^2}$

$$= \frac{\sqrt{17}}{2}$$

Area ΔOAC = Area ΔOBC $= \frac{1}{2}\sqrt{17}$

$$b-a = \begin{pmatrix}2\\1\\2\end{pmatrix} - \begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$c - a = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$(b-a) \times (c-a) = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{Area } \Delta ABC = \frac{1}{2}\left|(b-a) \times (c-a)\right|$$
$$= \frac{1}{2}\sqrt{1^2 + 1^2 + 1^2}$$
$$= \frac{\sqrt{3}}{2}$$

$$\therefore \text{ surface area of } T = 3 \times \frac{\sqrt{17}}{2} + \frac{\sqrt{3}}{2}$$
$$= \frac{1}{2}\sqrt{3}\left(\sqrt{3}\sqrt{17} + 1\right)$$
$$= \frac{1}{2}\sqrt{3}\left(\sqrt{51} + 1\right)$$
(as required)

6 **(a)** Determine all values of x for which $16x \equiv 5 \pmod{101}$. **[4]**

(b) Solve

(i) $95x \equiv 6 \pmod{101}$, **[2]**

(ii) $95x \equiv 5 \pmod{101}$. **[2]**

a) (working mod 101 throughout)

$$16x \equiv 5 \equiv 106 \ldots$$
$$\equiv 1520$$

we can divide by 16 since $(16,101)=1$

$\frac{1520}{16} = 95$

$\Rightarrow x = 95 \pmod{101}$

or

$x = 101n + 95$

bi) $95x \equiv 6$

$-6x \equiv 6$

$x \equiv -1 \pmod{101}$

bii) $95 \times 16 = 5 \pmod{101}$

$x = 16 \bmod (101)$

7 You are given the set $S = \{1, 5, 7, 11, 13, 17\}$ together with $\times_{18}$, the operation of multiplication modulo 18.

(a) Complete the Cayley table for $(S, \times_{18})$ given in the Printed Answer Booklet. [4]

(b) Prove that $(S, \times_{18})$ is a group. (You may assume that $\times_{18}$ is associative.) [3]

a)

	1	5	7	11	13	17
1	1	5	7	11	13	17
5	5	7	17	1	11	13
7	7	17	13	5	1	11
11	11	1	5	13	17	7
13	13	11	1	17	7	5
17	17	13	11	7	5	1

b) Identity is 1

Inverses: (5, 11) and (7, 13) are inverse-pairs; 17 is self-inverse

(c) Write down the order of each element of the group. **[2]**

(d) Show that $(S, \times_{18})$ is a cyclic group. **[1]**

(e) **(i)** Give an example of a non-cyclic group of order 6. **[1]**

(ii) Give one reason why your example is structurally different to $(S, \times_{18})$. **[1]**

Turn over for question 8

c) Elements: 1 5 7 11 13 17

Orders: 1 6 3 6 3 2

d) It has at least one element of order 6.

e i) • S_3, the group of 6 permutations of 3 symbols.

or

• D_3, the dihedral group of symmetries of the triangle.

or

• The product group $Z_3 \times Z_2$

e ii) • The non-cyclic group has elements of orders 1, 2, 2, 2, 3, 3

or • Noting that all elements have order 2 or 3.

or • This group is not abelian.

8 The motion of two remote controlled helicopters P and Q is modelled as two points moving along straight lines.

Helicopter P moves on the line $\mathbf{r} = \begin{pmatrix} 2+4p \\ -3+p \\ 1+3p \end{pmatrix}$ and helicopter Q moves on the line $\mathbf{r} = \begin{pmatrix} 5+8q \\ 2+q \\ 5+4q \end{pmatrix}$.

The function z denotes $(PQ)^2$, the square of the distance between P and Q.

(a) Show that $z = 26p^2 + 81q^2 - 90pq - 58p + 90q + 50$. **[3]**

(b) Use partial differentiation to find the values of p and q for which z has a stationary point. **[4]**

(c) With the aid of a diagram, explain why this stationary point must be a minimum point, rather than a maximum point or a saddle point. **[2]**

(d) Hence find the shortest possible distance between the two helicopters. **[2]**

The model is now refined by modelling each helicopter as a sphere of radius 0.5 units.

(e) Explain how this will change your answer to part **(d)**. **[2]**

a) $\vec{PQ} = \begin{pmatrix} 5+8q \\ 2+q \\ 5+4q \end{pmatrix} - \begin{pmatrix} 2+4p \\ -3+p \\ 1+3p \end{pmatrix} = \begin{pmatrix} 3+8q-4p \\ 5+q-p \\ 4+4q-3p \end{pmatrix}$

$z = (PQ)^2 = (3+8q-4p)^2 + (5+q-p)^2 + (4+4q-3p)^2$

$= (9 + 24q - 12p + 24q + 64q^2 - 32pq - 12p - 32pq + 16p^2)$

$+ (25 + 5q - 5p + 5q + p^2 - pq - 5p - pq + q^2)$

$+ (16 + 16q - 12p + 16q + 16q^2 - 12pq - 12p - 12pq + 9p^2)$

$= 81q^2 + 26p^2 + 50 - 90pq - 58p + 90q$

$= 26p^2 + 81q^2 - 90pq - 58p + 90q + 50$

(as required)

b) $\frac{dz}{dp} = 52p - 90q - 58$

$0 = 52p - 90q - 58$

$58 = 52p - 90q$

$29 = 26p - 45q$ — ①

$\frac{dz}{dq} = 162q - 90p + 90$

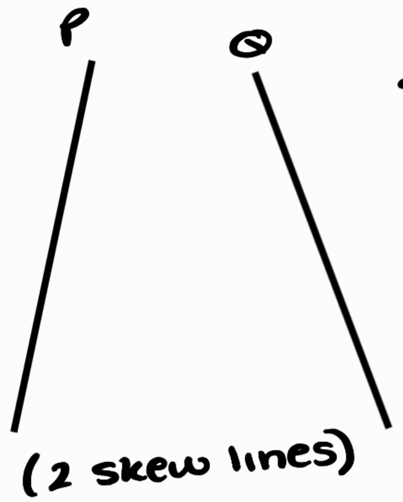
$0 = 162q - 90p + 90$

$90 = 90p - 162q$

$45 = 45p - 81q$ — ②

$p = 4, \quad q = \frac{5}{3}$

c)



- Moving P and Q in 'opposite' directions along their lines gives Z indefinitely large, hence stationary point is not a maximum.
- Symmetric properties of P, Q gives both max or both min so not a saddle-point.

d) Substituting $p=4$, $q=\frac{5}{3}$ into

$Z = 26p^2 + 81q^2 - 90pq - 58p + 90q + 50$

$= 9$

$\therefore$ shortest distance $= \sqrt{9} = 3m$

e) Because they are modelled as spheres, for any value of p and q the distance between them will simply be less than the original model. The shortest distance is now $3 - 1 = 2m$.